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Green functions for topology change

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Abstract: We explicitly calculate the Green functions describing quantum changes of topology in Friedman-Lemaître-Robertson-Walker universes the spacelike sections of which are compact but endowed with distinct topologies. The calculations are performed using the long wavelength approximation at second order in the gradient expansion. We argue that complex metrics are necessary in order to obtain a non-vanishing Green functions and interpret this fact as demonstrating that a quantum topology change can be viewed as a quantum tunneling effect. We demonstrate that quantum topological transitions between curved hypersurfaces are allowed whereas no transition to or from a flat section is possible, establishing thus a selection rule. We also show that the quantum topology changes in the direction of negatively curved hypersurfaces are strongly enhanced as time goes on, while transitions in the opposite direction are suppressed.

Keywords: Models of Quantum Gravity, Cosmology of Theories beyond the SM.

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Contents

1. Introduction

Combining the principles of General Relativity with those of Quantum Mechanics seems necessary in order to better understand the physical nature of the gravitational field. As a result, new properties are expected to emerge. For instance, in classical General Relativity, topology changes are forbidden in the sense that their presence would necessarily imply the appearance of either singularities or closed timelike curves, a result known as the Geroch theorem, see ref. [1]. On the other hand, topology changes are widely believed to become allowed if the quantum-mechanical nature of the gravitational field is taken into account. Studying this problem is notoriously known as a difficult technical task. In the literature, various aspects of this question have been discussed. For instance, general properties have been studied in refs. [2, 3] while applications to black holes physics and to cosmology have been investigated in refs. [4, 5, 6] and [7, 8] respectively. Recently, a model where the theoretical ideas about topology change can be implemented in cosmology, at the level of equations, has been proposed in ref. [9]. Concretely, the metric of the model put forward in ref. [9] is given by a Friedmann-Lemaître-Robertson-Walker-like metric where the curvature of the spacelike sections, usually denoted as $\mathcal{K}$, is now a time dependent function, namely $\mathcal{K} = \mathcal{K}(t)$,

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\left\{\mathrm{d}\xi^2 + \left[\frac{\sin(\sqrt{\mathcal{K}}\xi)}{\sqrt{\mathcal{K}}}\right]^2 \mathrm{d}\Omega^2(\theta,\varphi)\right\}. \tag{1.1}$$

The three-dimensional spacelike hypersurfaces $\mathbb{V}^3$ of the model may be endowed with a large class of topologies compatible with such geometry. Technically, the metric has always the form displayed above (since it is locally defined) but the ranges of variations of the coordinates are modified. The scalar curvature of the metric is always constant (for fixed t) and equal to $6\mathcal{K}/a^2$. In ref. [9], we have restricted our considerations to compact (and orientable) spaces in order to avoid possible surface terms in the hamiltonian formalism and we will make the same assumption in the present article. We can show that any compact three manifold with constant curvature is homeomorphic to $\mathbb{V}^3/\Gamma$, where the universal covering $\mathbb{V}^3$ is either $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, according to the sign of $\mathcal{K}$ ($\mathcal{K} = 0$, $\mathcal{K} < 0$, $\mathcal{K} > 0$, respectively). The group Γ is the group of covering transformations. In three dimensions the closed (i.e. compact without boundary) space is now homeomorphic to a polyhedron the faces of which are identified by pairs, for a general review see ref. [10]. Let us now discuss in more details the three possible cases.

Let us first consider the case $\mathcal{K} > 0$. The universal covering is $\mathbb{S}^3$. Since it is compact, all the three-dimensional spaces admitting this universal covering are also compact. There is an infinite number of such spaces. We will only consider two cases: $\mathbb{S}^3$ itself and the Poincaré dodecahedral space, $\mathbb{D}^3 \equiv \mathbb{S}^3/I^*$, where I^* is the binary symmetry group of the icosahedron. The ranges of variation of the coordinates (ξ,θ,φ) can always be written as

$$0 \leq \xi \leq \frac{\Xi(\theta, \varphi; \mathbb{V}^3)}{\sqrt{\mathcal{K}}}, \qquad 0 \leq \theta \leq \pi, \qquad 0 \leq \varphi \leq \pi. \tag{1.2}$$

The symbol $\mathbb{V}^3$ in the argument of the function Ξ indicates that $\Xi(\theta, \varphi; \mathbb{V}^3)$ is not the same function for different $\mathbb{V}^3$ ($\mathbb{S}^3$ or $\mathbb{D}^3$). Of course, the case $\mathbb{S}^3$ is very simple and we just have $\Xi(\theta, \varphi; \mathbb{S}^3) = \pi$. This guarantees that no conical singularities will appear in this case. If $\mathbb{V}^3 = \mathbb{D}^3$ the function Ξ is also not arbitrary but its explicit expression is much more complicated. For the case $\mathcal{K} < 0$, where the universal covering is $\mathbb{H}^3$, the classification of closed three-dimensional spaces is still an open question, but it is known that the volumes of the polyhedron are fixed as in the $\mathcal{K} > 0$ case. Finally, the case $\mathcal{K} = 0$ have six different tessellations by polyhedrons, some of them with arbitrary volume for their polyhedrons (for more details see again ref. [10]). What is important for us here is that the topologies compatible with one $\mathcal{K}$ are not compatible with other $\mathcal{K}$ of distinct sign. Hence, a change in the sign of $\mathcal{K}$, or in the sign of the curvature scalar $6\mathcal{K}/a^2$, necessarily indicates a change of topology. Our strategy here will be to investigate whether this change is possible quantum mechanically.

The quantization of this model has been carried out in ref. [9]. In that article, it has been shown that a hamiltonian formulation necessarily requires a passage to a *midisuperspace* description. Indeed, it is not possible to construct a *minisuperspace* hamiltonian from the metric given in eq. (1.1) because, if $\mathcal{K}$ depends on time, this metric does not represent a spatially homogeneous space-time, in the sense that the components of the four dimensional curvature tensor in a local frame are functions of t and ξ. The existence of the non-vanishing component of the Ricci tensor, $R_{t\xi} = [\xi\dot{\mathcal{K}}(t)]/[a(t)N(t)] \neq 0$, is a consequence of this fact. Hence, in ref. [9], we have been forced to introduce a *midisuperspace* model having a non-vanishing shift function $N_\xi(\xi,t)$. As a consequence, the metric has been written as

$$\mathrm{d}s^2=\left[-N^2(\xi,t)+\frac{N_\xi^2(\xi,t)}{a^2(\xi,t)}\right]\mathrm{d}t^2+2N_\xi(\xi,t)\mathrm{d}\xi\mathrm{d}t+a^2(\xi,t)\left[\mathrm{d}\xi^2+\sigma^2(\xi,t)\mathrm{d}\Omega^2(\theta,\varphi)\right]. \quad (1.3)$$

The first step has been to carry out the quantization of this *midisuperspace* model. Then, in a second step, we have taken into account in the quantum solutions the restrictions on the variables a and σ which, from eq. (1.3), allows us to recover the metric given in eq. (1.1). Consistency requires that we first treat the *midisuperspace* problem in order to come back to the *minisuperspace* model afterward. Correspondingly, the Wheeler-DeWitt equation of the model remains a functional differential equation rendering the findings of general exact solutions problematic. In ref. [9], only two particular semi-classical solutions have been found for which the possibility of a topology change at the quantum level has been explicitly demonstrated.

The present article aims at investigating the same problem, i.e. whether topology changes are possible quantum-mechanically, but with the help of a different method. We will study the transition between an initial and a final configurations consisting in two FLRW metrics with fixed (constant) spatial curvature but with different signs and will calculate the Green functions which completely characterize the quantum evolution of the system. For a point particle, the Green function $G(x_\mathrm{f},t_\mathrm{f};x_\mathrm{i},t_\mathrm{i})$ represents the probability amplitude to find the particle at point x_f and time t_f, knowing that it was at point x_i at the initial time t_i. In fact, it is necessary to smooth out the Green function because, defined as before, it is not square integrable. Physically, this is due to the fact that the particle can never be localized exactly. In the present context, a Green function can also be defined and a similar interpretation can be made. In this case, the wave function depends on the volume of the spacelike sections $y\propto a^3$, on the three curvature R, and on a dust field $\chi(t)$ (rigorous definitions are given below), which plays the role of time: $\Psi=\Psi(y,R,\chi)$, see ref. [11]. The Green function $G(y_\mathrm{f},R_\mathrm{f},\chi_\mathrm{f};y_\mathrm{i},R_\mathrm{i},\chi_\mathrm{i})$ can now be defined as the probability amplitude of having a volume y_f and a three-curvature R_f at "time" χ_f knowing that, at initial χ_i, the space volume was y_i and the three-curvature R_i. Therefore, the Green function defined above represents the amplitude of probability for a topology transition as soon as R_f has not the same sign as R_i. This is the quantity that we would like to calculate.

Technically, there are at least two methods which permit to calculate a Green function. The first one simply uses its definition, namely

$$\Psi(q_\mathrm{f},t_\mathrm{f})=\int\mathrm{d}q_\mathrm{i}G(q_\mathrm{f},t_\mathrm{f};q_\mathrm{i},t_\mathrm{i})\Psi(q_\mathrm{i},t_\mathrm{i})\,, \quad (1.4)$$

where, in the above formula, q denotes the degrees of freedom of the physical system under consideration. Then, the method consists in manipulating this expression in order to isolate the Green function and express it as a sum over quantities characterizing the possible states of the system. For instance, in the case of a point particle with a time independent hamiltonian, the Green function can be written as a weighted sum over the product of the eigenstates, see e.g. ref. [12],

$$G(x_{\rm f},t_{\rm f};x_{\rm i},t_{\rm i})=\sum_n \varphi_n^*(x_{\rm i})\varphi_n(x_{\rm f}){\rm e}^{-iE_n(t_{\rm f}-t_{\rm i})/\hbar}\,. \tag{1.5}$$

This is the method that will be used here and the main formula of this article, eqs. (3.9) and (4.20), will be similar to eq. (1.5).

Another method, different from the previous one, is based on the path integral formalism of quantum mechanics. In this approach, the propagator is given by a sum over all the paths connecting the initial and final configurations in the space of metrics. As there are no real classical paths connecting configurations with different topologies, even a semi-classical calculation would require to find paths in the space of complex geometries. It is worth pointing out here that the use of complex metrics has already been considered in similar problems, see for instance ref. [7], although the context of this last reference was different. In fact, we will show in the present article that the use of complex metrics is in fact equivalent to the use of complex paths routinely considered in ordinary quantum mechanics. An important consequence for our study is that the path connecting the initial and final configurations does not necessarily go through real lorentzian geometries. For instance, if the initial configuration is given by a metric (1.1) with $\mathcal{K}=1$ and the final configuration is given by the same metric but this time with $\mathcal{K}=-1$, then it is clear that a complex path connecting the two does not necessarily pass through the "point" representing a metric (1.1) with $\mathcal{K}=0$.

However, using the path integral formalism in order to determine the Green function would require the calculation of all the complex paths connecting the initial and the final configurations which, in turn, would necessitate to classify all the possible topologies between them. Clearly, this would be extremely complicated and is probably beyond the present mathematical knowledge. This is why, in this paper, we will use the first method only and not the one based on the path integral formalism. Of course, in standard quantum mechanics, the two approaches are strictly equivalent and it is only a matter of taste to use one rather than the other. In the present context this should be true as well, although the proof of this assertion is obviously much more complicated. A full investigation of this issue is beyond the scope of the present paper. Although we will not use the path integral formalism, the idea that two configurations of different topologies can be connected by a complex path which does not necessarily pass through a real lorentzian metric will turn out to be very useful. In particular, in the following, we will use explicitly the fact that the metric can be complex.

Even with the first method, a full calculation of the Green functions is probably beyond our present computational capability and, therefore, we will be forced to rely on some approximation methods. Firstly, we will restrict ourselves to a semi-classical approximation of the Wheeler-De Witt equation. In this framework, the wave function will be given by the semi-classical wave function $\Psi=\exp(iS/\hbar)$. Secondly, as we are interested in large scale quantum changes of topology in the homogeneous background and not on the small scale ones, typical of Wheeler's space-time foam, we will use the long-wavelength approximation, developed in refs. [13]–[16], which is valid whenever the characteristic scale of the problem is

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much bigger than the Hubble radius, and permits a complete analysis of the semi-classical wave function and Green functions in the full superspace. Then, we will show how, by generalizing a method used in ref. [16], the Green function can be isolated yielding an equation analogous to eq. (1.5). We will exhibit explicit analytical expressions for the Green function, some of them signaling a topology change. As already discussed, although the transition occurs along a path in the space of complex metrics, we will not need to specify this path. Also, we will demonstrate that, for some transitions, the corresponding Green function vanishes, establishing thus selection rules for topology changes.

This article is organized as follows. In the second section, we briefly review the long wavelength approximation and the evaluation of the Hamilton-Jacobi function up to the second order (first order in the curvature). Then, in the third section, we perform the calculations in the case of real metrics and show that there is no quantum change of topology. For completeness, we also present a generalization of the results obtained in ref. [15]. In the fourth section, we follow the discussion presented in the Introduction and argue that complex metrics are necessary in order to obtain quantum changes of topology. Then, we investigate the corresponding modifications to the hamiltonian formalism. Finally, as outlined before, we explicitly calculate the corresponding Green functions, using the method presented previously, without relying on the path integral approach and we also discuss the main consequences that can be drawn from their expressions. In the last and fifth section we present our conclusions.

2. The long wavelength approximation for quantum gravity with dust

In this section, we discuss the long-wavelength approximation technique used to solve the hamiltonian-Jacobi equation which is nothing but the equation for the phase functional in a Wentzel-Kramer-Brillouin (WKB) semi-classical approximation to canonical quantum gravity. This technique consists in the expansion of the phase functional (or the generating functional in a hamiltonian formulation of General Relativity) in a series of spatial gradients and was introduced by Salopek, Stewart and Parry [13, 14, 15] as a method for solving the full Hamilton-Jacobi equation for gravity interacting with matter. In ref. [15] these authors were able to exhibit a recurrence formula from which they could obtain solutions to the Hamilton-Jacobi equation to any order of approximation for matter made of scalar and dust fields. This will be crucial for our treatment because we shall use the second order solution to obtain the Green functions associated with the WKB wave functional. As we will show, the solution in the second order approximation will allow us to describe changes of topology because the terms in the second order expansion of the WKB phase functional explicitly contain spatial curvature terms. As already mentioned, changes in the sign of these terms indicate a change of topology.

Let us now briefly describe the formalism introduced in refs. [13, 14, 15]. The action for General Relativity interacting with a dust field χ is

$$S = \int \mathrm{d}^4x \sqrt{-g}\left[\frac{1}{2\kappa}{}^{(4)}R - \frac{n}{2m}\left(g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + m^2\right) - V(\chi)\right], \tag{2.1}$$

where $\kappa \equiv 8\pi/m_{\rm Pl}^2$, $m_{\rm Pl}$ being the Planck mass. ${}^{(4)}R$ is the Ricci scalar of the space-time metric $g_{\mu\nu}$, and χ is a velocity potential for irrotational dust particles with rest mass m. $V(\chi)$ is a potential for the dust field χ and n is the rest number density of the dust particles. The dust field χ defines the four velocity field for dust particles by

$$u^\mu = -g^{\mu\nu}\frac{1}{m}\chi_{,\nu} \tag{2.2}$$

so that $\chi =$ const. will determine a congruence of spacelike hypersurfaces foliating the space-time (notice that $g^{\mu\nu}\chi_{,\mu}\chi_{,\nu} = -m^2$). Therefore, the dust field can be used as the time variable for our model, as well as in the Schrödinger-type equation obtained from the Wheeler-DeWitt equation.

In the Arnowitt-Deser-Misner formalism the line element reads

$$\mathrm{d}s^2 = (-N^2 + \gamma^{ij}N_iN_j)\mathrm{d}t^2 + 2N_i\,\mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^i\,, \tag{2.3}$$

where N and N_i are the lapse and shift functions, respectively, and γ_{ij} is the three-metric of the spacelike hypersurface. Then, the action (2.1) can be rewritten as

$$S = \int \mathrm{d}^4x(\pi^\chi\dot{\chi} + \pi^{ij}\dot{\gamma}_{ij} - N\mathcal{H} - N^i\mathcal{H}_i)\,, \tag{2.4}$$

where π^{ij} are the momenta conjugate to γ_{ij} and $\pi^\chi = n\gamma^{1/2}(1 + \chi_{,i}\chi^{,i}/m^2)^{1/2}$ is the momentum conjugate to the dust field χ. The quantities $\mathcal{H}$, $\mathcal{H}_i$ are given by the following expressions

$$\mathcal{H} = \kappa\gamma^{-1/2}\pi^{ij}\pi^{k\ell}(2\gamma_{jk}\gamma_{\ell i} - \gamma_{ij}\gamma_{k\ell}) - \frac{1}{2\kappa}\gamma^{1/2}R + (m^2 + \chi_{,i}\chi^{,i})^{1/2}\pi^\chi + \gamma^{1/2}V(\chi)\,, \tag{2.5}$$

$$\mathcal{H}_i = -2(\gamma_{i\ell}\pi^{\ell k})_{,k} + \pi^{\ell k}\gamma_{\ell k,i} + \pi^\chi\chi_{,i}\,, \tag{2.6}$$

where R is the Ricci scalar of the three-metric γ_{ij}.

At this point it is well worth making some dimensional analysis, in order to connect the conditions of validity of the long-wavelength approximation to the Hubble radius of the Universe. From eq. (2.5), we can see that the long-wavelength approximation implies that the first term on the right-hand side is much larger than the second term, at first order in the approximation. From the expression for $\pi^{ij} = (K^{ij} - h^{ij}K)/\kappa$, see ref. [9], we can see that π^{ij} has the dimension L^{-3} (in the system of units where $\hbar = c = 1$) where L stands for a length. More explicitly,

$$\pi^{ij} \sim \frac{1}{\ell_{\rm Pl}^2\ell_{\rm H}}\,,$$

where $\ell_{\rm Pl} = 1/m_{\rm Pl}$ is the Planck length and $\ell_{\rm H}$ is essentially the Hubble radius. Therefore, the validity of the long-wavelength approximation imposes that, see eq. (2.5),

$$\lambda \gg \ell_{\rm H}\,,$$

where the wavelength λ is a typical length associated with the scales where spatial gradients are relevant. In other words, λ must be greater than the characteristic Hubble radius of

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the model. In our scheme, topology change is a large-scale (global) phenomenon and the long-wavelength approximation is therefore the appropriate approximation to probe it. The above condition, obviously, should not contain the Planck length because eq. (2.5) is a classical equation. Furthermore, in principle, there is no problem in applying this condition to closed models, even when their volumes becomes close to the Planck volume, (where, anyway, our semi-classical approximation should not be valid) if the Hubble radius at this epoch is smaller than the Planck length.

At the quantum level the above system may be quantized following the Dirac's prescription [17]. The super-hamiltonian and super-momentum constraints given by eqs. (2.5) and (2.6) become operators and, when applied to the wave functional of the system, result in two relations which express that only a restricted region of the Hilbert space of wave functionals contains the physical states of the theory:

$$\mathcal{H}\Psi = 0\,, \qquad \mathcal{H}_i\Psi = 0\,. \tag{2.7}$$

The first equation is the well-known Wheeler-DeWitt equation whereas the second one is the so-called quantum momentum constraint. In what follows, our treatment of quantum changes of topology will be done at the level of the semi-classical approximation. This means that the wave functionals are assumed to be of the WKB form, namely

$$\Psi = \mathrm{e}^{iS/\hbar}\,, \tag{2.8}$$

where S is the action. At order $\hbar^0$, eqs. (2.5) and (2.6) reduce to the Hamilton-Jacobi functional equation

$$\begin{aligned}\mathcal{H}(x) = \gamma^{-1/2}\kappa\,\frac{\delta S}{\delta\gamma_{ij}(x)}\,\frac{\delta S}{\delta\gamma_{k\ell}(x)}[2\gamma_{i\ell}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{k\ell}(x)] +\\ +\sqrt{m^2+\gamma^{ij}\chi_{,i}\chi_{,j}}\,\frac{\delta S}{\delta\chi(x)} + \gamma^{1/2}\,V(\chi) - \frac{1}{2\kappa}\,\gamma^{1/2}R = 0\,,\end{aligned} \tag{2.9}$$

and to the momentum constraint equation

$$\mathcal{H}_i(x) = -2\left[\gamma_{ik}\,\frac{\delta S}{\delta\gamma_{kj}(x)}\right]_{,j} + \frac{\delta S}{\delta\gamma_{\ell k}(x)}\,\gamma_{\ell k,i} + \frac{\delta S}{\delta\chi}\,\chi_{,j} = 0\,, \tag{2.10}$$

the solution of which is the phase S of the WKB wave functional, see eq. (2.8). In ref. [15], Parry, Salopek and Stewart have been able to derive solutions for these equations using the long wavelength approximation. The method is based on the so-called spatial gradient expansion of the phase functional S. It consists in expanding S in a series of terms according to the number of spatial gradients they contain, namely $S = S^{(0)} + S^{(2)} + S^{(4)} + \cdots$. As a consequence, the Hamilton-Jacobi equation can be grouped in terms with an even number of spatial derivatives $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \mathcal{H}^{(4)} + \cdots$. At zeroth order, the Hamilton-Jacobi equation reduces to

$$\mathcal{H}^{(0)} = \gamma^{-1/2}\kappa\,\frac{\delta S^{(0)}}{\delta\gamma_{ij}}\,\frac{\delta S^{(0)}}{\delta\gamma_{k\ell}}(2\gamma_{jk}\gamma_{\ell i} - \gamma_{ij}\gamma_{k\ell}) + m\frac{\delta S^{(0)}}{\delta\chi} + \gamma^{1/2}V(\chi) = 0\,. \tag{2.11}$$

For a dust field χ with a vanishing potential, $V(\chi) = 0$, the invariance under diffeomorphisms of the generating functional suggests a solution of the form

$$S^{(0)} = -\frac{2}{\kappa}\int \mathrm{d}^3x\gamma^{1/2}H(\chi)\,. \tag{2.12}$$

This functional satisfies the Hamilton-Jacobi relation if $H(\chi)$ is a solution of the equation $H^2 = -(2m/3)\partial H/\partial\chi$, yielding

$$H(\chi) = \frac{2m}{3(\chi - \tilde{\chi})}\,, \tag{2.13}$$

where $\tilde{\chi}$ is an integration constant. The functional $S^{(0)}$ is clearly invariant under coordinate transformations and, therefore, satisfies eq. (2.10). In addition, it contains no spatial derivatives. For the class of generating functionals given by eq. (2.12), the second-order Hamilton-Jacobi equation reads

$$\mathcal{H}^{(2)} = 2H\gamma_{ij}\frac{\delta S^{(2)}}{\delta\gamma_{ij}} + m\frac{\delta S^{(2)}}{\delta\chi} + \frac{1}{2m}\gamma^{ij}\chi_{,i}\chi_{,j}\frac{\delta S^{(0)}}{\delta\chi} - \frac{1}{2\kappa}\gamma^{1/2}R = 0\,. \tag{2.14}$$

In ref. [15], Parry, Salopek and Stewart obtained a diffeomorphisms invariant solution for $S^{(2)}$ which can be written as

$$S^{(2)} = \frac{1}{\kappa}\int \mathrm{d}^3x\gamma^{1/2}J(\chi)R\,, \tag{2.15}$$

where the function $J(\chi)$ is obtained by substituting eq. (2.15) into eq. (2.14) and using the zeroth order solution (2.12). This leads to

$$HJ + m\frac{\partial J}{\partial\chi} = \frac{1}{2}\,, \tag{2.16}$$

which yields for $S^{(2)}$

$$S^{(2)} = \frac{1}{\kappa}\int \mathrm{d}^3x\,\gamma^{1/2}\left[\frac{3}{10m}(\chi - \tilde{\chi}) + D(\chi - \tilde{\chi})^{-2/3}\right]R\,, \tag{2.17}$$

where D is an arbitrary integration constant. We notice the fact that $S^{(2)}$ contains the three-curvature term R whereas this is not the case for $S^{(0)}$. Since a topology change is signaled by a change of sign of the three-curvature R, it is necessary to push the long wavelength expansion to the second order in order to have non trivial effects. In other words, the quantum topology changes can only be studied at this (second) order.

3. The Green function with real metrics

3.1 General derivation

Let us now consider the semi-classical wave function up to the second order in the long wavelength approximation $\Psi = \exp\{i[S^{(0)} + S^{(2)}]/\hbar\}$ with $S^{(0)}$ and $S^{(2)}$ given by eqs. (2.12) and (2.17), as explained in the previous section. Since we restrict ourselves to geometries of the FLRW form, the integrals in eqs. (2.12) and (2.17) can be performed exactly yielding

$$\Psi(y,R,\chi) = \exp\left\{\frac{i}{\hbar\kappa}\left[-2yH + \frac{3yR(\chi-\tilde{\chi})}{10m} + \frac{yRD}{(\chi-\tilde{\chi})^{2/3}}\right]\right\}, \tag{3.1}$$

where the quantity y given by

$$y \equiv \int \mathrm{d}^3x\gamma^{1/2} = Va^3 \tag{3.2}$$

is the total volume associated with the metric (1.1). V is the constant volume of the cell corresponding to the homogeneous polyhedron of the tessellation of the spatial sections that are compactified. As previously, H can be expressed as $H = 2m/[3(\chi-\tilde{\chi})]$ and R is the three-curvature which can be written as $R = 6\mathcal{K}/a^2$. Since V may have a discontinuous jump while R' goes through zero, the Green function is defined as

$$\begin{aligned}\Psi(y,R,\chi) \equiv & \int_0^\infty \mathrm{d}y' \int_0^\infty \mathrm{d}R' G(y,R,\chi;y',R',\chi')\Psi(y',R',\chi') + \\ & + \int_0^\infty \mathrm{d}y' \int_{-\infty}^0 \mathrm{d}R' G(y,R,\chi;y',R',\chi')\Psi(y',R',\chi') + \\ & + \int_0^\infty \mathrm{d}y' G(y,R,\chi;y',0,\chi')\Psi(y',\chi')\,. \end{aligned}\tag{3.3}$$

Our goal is now to establish the expression for the Green function itself from the above equation. In this approach, the integration is only performed over the quantities characterizing the initial metric. So the procedure is as follows: we use the initial field configuration $\Psi(y',R',\chi')$ and the final field configuration $\Psi(y,R,\chi)$, which are both solutions of the dynamical equations and then find a propagator that connects one to the other. We know from ref. [15] how to utilize eq. (3.3) in order to extract the propagator. In the following, we will just consider a generalization of this method.

For the purpose of calculating the Green function from eq. (3.3), it is convenient to use a new variable v, which is a combination of y and R, namely $v \equiv yR$. To obtain the Green function, we use the fact that eq. (3.3) is valid for any value of the integration constants $\tilde{\chi}$ and D. However, instead of the integration constants $\tilde{\chi}$ and D, it will be more appropriate to work with new integration constants defined by the following relations (for fixed χ')

$$H'(\tilde{\chi}) \equiv \frac{2m}{3(\chi'-\tilde{\chi})}, \qquad J'(\tilde{\chi},D) \equiv \frac{3}{10m}(\chi'-\tilde{\chi}) + D(\chi'-\tilde{\chi})^{-2/3}. \tag{3.4}$$

We have just replaced the two constants $(\tilde{\chi},D)$ by two new constants (H',J'). Let us notice that J' is in fact $J(\chi')$, see eqs. (2.15) and (2.17), hence its name. If we inverse the above relations, one arrives at

$$\tilde{\chi}(H') = \chi' - \frac{2m}{3H'}, \tag{3.5}$$

$$D(H',J') = J'\left(\frac{2m}{3H'}\right)^{2/3} - \frac{3}{10m}\left(\frac{2m}{3H'}\right)^{5/3}. \tag{3.6}$$

Let us stress again that these expressions are defined for a fixed value of χ'. With these new definitions and using the explicit expression of the wave function given in eq. (3.1), eq. (3.3) now reads

$$\begin{aligned}
&\exp\left(\frac{i}{\hbar\kappa}\left\{-\frac{4my}{3[\chi-\tilde{\chi}(H')]}+\frac{3v[\chi-\tilde{\chi}(H')]}{10m}+\frac{vD(H',J')}{[\chi-\tilde{\chi}(H')]^{2/3}}\right\}\right)=\\
&=\int_0^\infty \frac{\mathrm{d}y'}{y'}\int_0^\infty \mathrm{d}v' G\left(y,\frac{v}{y},\chi;y',\frac{v'}{y'},\chi'\right)\exp\left[\frac{i}{\hbar\kappa}(-2y'H'+v'J')\right]+\\
&+\int_0^\infty \frac{\mathrm{d}y'}{y'}\int_{-\infty}^0 \mathrm{d}v' G\left(y,\frac{v}{y},\chi;y',\frac{v'}{y'},\chi'\right)\exp\left[\frac{i}{\hbar\kappa}(-2y'H'+v'J')\right]+\\
&+\int_0^\infty \frac{\mathrm{d}y'}{y'} G\left(y,\frac{v}{y},\chi;y',0,\chi'\right)\exp\left[\frac{i}{\hbar\kappa}(-2y'H')\right]. \qquad (3.7)
\end{aligned}$$

We see the effect of having introduced the "new" integration constants: the left-hand-side of the above equation now depends on both χ and χ'. In order to extract $G(y,\, v/y,\, \chi; y',\, v'/y',\, \chi')$, we multiply eq. (3.7) by the factor

$$\exp\left[\frac{i}{\hbar\kappa}(2y''H'-v''J')\right], \qquad (3.8)$$

and integrate it over J' and H' from $-\infty$ to ∞. The result of this integration on the right-hand-side of eq. (3.7) is very simple. The integration over H' yields a Dirac delta function $\delta(y''-y')$ which makes the integration over y' in the right-hand-side of eq. (3.7) very simple. If $v''>0$ ($v''<0$), then only the first (second) integral over J' in the right-hand-side of eq. (3.7) does not vanish, leading, after integration over v', to

$$\begin{aligned}
G\left(y,\frac{v}{y},\chi;y'',\frac{v''}{y''},\chi'\right) &= \frac{y''}{2\pi^2\hbar^2\kappa^2}\int_{-\infty}^\infty \mathrm{d}H'\int_{-\infty}^\infty \mathrm{d}J'\Psi(y,v,\chi)\Psi^*(y'',v'',\chi')\\
&=\frac{y''}{2\pi^2\hbar^2\kappa^2}\int_{-\infty}^\infty \mathrm{d}H'\int_{-\infty}^\infty \mathrm{d}J'\times\\
&\quad\times\exp\left\{\frac{i}{\hbar\kappa}\left[-\frac{2ybH'}{H'+b}+2y''H'-v''J'+\right.\right. \qquad (3.9)\\
&\qquad\left.\left.+v\left(J'-\frac{1}{5H'}\right)\left(\frac{b}{H'+b}\right)^{2/3}+\frac{v}{5}\left(\frac{1}{b}+\frac{1}{H'}\right)\right]\right\},
\end{aligned}$$

where $b\equiv 2m/[3(\chi-\chi')]$. The first term in the argument of the exponential has been obtained by using the relation $H=2m/[3(\chi-\tilde{\chi})]$ and then eqs. (3.5) and (3.6). The two last terms have been obtained by using the explicit expressions of $\tilde{\chi}$ and D, see eqs. (3.5) and (3.6). In the same manner, if $v''=0$ then only the last integral over J' in the right-hand-side of eq. (3.7) may not vanish

$$\begin{aligned}
G\left(y,\frac{v}{y},\chi;y'',0,\chi'\right) &= \frac{1}{\delta(0)}\frac{y''}{2\pi^2\hbar^2\kappa^2}\int_{-\infty}^\infty \mathrm{d}H'\int_{-\infty}^\infty \mathrm{d}J'\times\\
&\quad\times\exp\left\{\frac{i}{\hbar\kappa}\left[-\frac{2ybH'}{H'+b}+2y''H'+v\left(J'-\frac{1}{5H'}\right)\left(\frac{b}{H'+b}\right)^{2/3}+\right.\right.
\end{aligned}$$

$$+\frac{v}{5}\left(\frac{1}{b}+\frac{1}{H'}\right)\Bigg]\Bigg\}. \tag{3.10}$$

The eqs. (3.7) and (3.10) are the main result of this section. Clearly, the central issue is now the calculation of the integrals over J' and H'. From eq. (3.9), we see that the integral over H' is to be performed along the real axis. However, in the next subsection, dealing with the case of flat hypersurfaces, the integration over H' will be performed in the complex plane, along a closed contour. It is important to notice that this procedure is just a technical trick, allowing us to simplify the calculation of the integral by the use of the Cauchy theorem, but it has no deep physical meaning. Of course, all the following complex contours turn out to be, after having taken the appropriate limits, equivalent to the integration along the real axis as given in eq. (3.9).

Let us also emphasize that, without using the path integral approach, we have isolated the Green function and put it under the form of an integral over the quantities characterizing the state of the system. Although the Green function in eq. (3.9), which is analogous to eq. (1.5) for the system considered in this paper, should also result from a path integral calculation with a sum performed over all the real paths connecting the initial and final configurations in the space of metrics, no such sum has been actually calculated. We will see in the next section that, in order to obtain topology changes, complex paths connecting the initial and final configurations will be necessary. However, the explicit characterization of those complex paths will be avoided by using the same type of manipulations as shown above.

3.2 The case of flat hypersurfaces

Firstly, let us consider the case $v''=0$. From the above considerations, it is clear that eq. (3.10) does not vanish only if $v=0$ since, in this case, the integration over J' produces a $\delta(0)$ which exactly cancels the one already present. This means that the three-curvature remains the same (i.e. zero) and there is obviously no topology change. Therefore, if $v''=0$ then this implies that $v=0$. Secondly, let us consider that $v=0$. In this case, from eq. (3.9), we see that the integration over J' is necessarily zero if $v''\neq 0$. As a consequence, $v=0$ implies that $v''=0$, leading to the conclusion that $v''=0$ is in fact equivalent to $v=0$.

At this point, however, we go further and depict the calculation of the Green function despite the fact that there is no change of topology and despite the fact that the final result is already known, see ref. [16]. The main reason for doing this is that this will exemplify techniques that will be used in this paper later on. This calculation is therefore a warm-up. As mentioned before, the integral in the right-hand-side of eq. (3.10) can be calculated by extending H' to the complex plane with the choice of the integration contour represented by the solid line in figure 1. This contour contains the pole $H'=-b$ of the exponential function appearing in the right-hand-side of eq. (3.10). The fact that $H'=-b$, which is a pole of the argument of the exponential, is also a pole of the exponential itself can easily be seen if one uses the Taylor expansion of the exponential function. When the limits $\rho\to+\infty$ and $\epsilon\to 0$ are taken, see figure 1, the integral over H' in the right-hand-side of

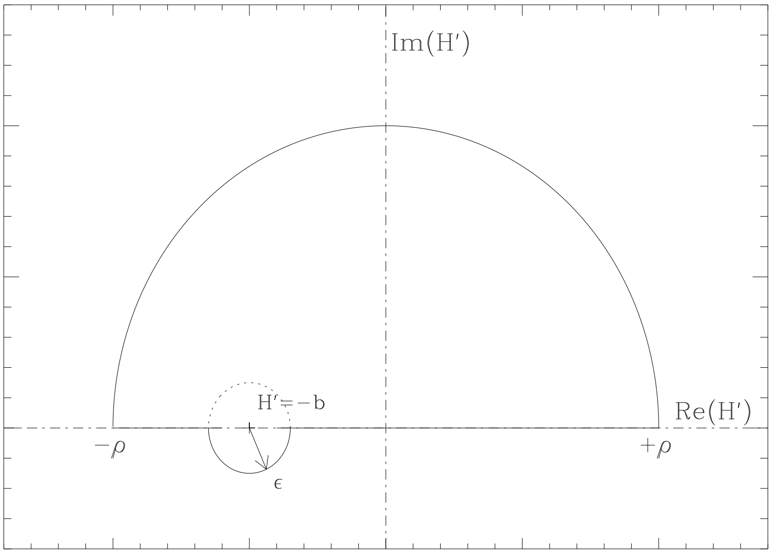

Figure 1: The contour of integration $\mathcal{C}$ in the complex plane $[\Re(H'), \Im(H')]$. The contour chosen in this article is the solid line contour. Because of the Cauchy's theorem, $\mathcal{C}$ is equivalent to the small circle contour centered at $H' = -b$.

eq. (3.10) reduces to an integration on the real axis, the contribution coming from the big semicircle in the upper-half complex plane going to zero as a consequence of the Jordan's lemma. Because of the Cauchy's theorem, the integration along $\mathcal{C}$ is in fact equivalent to the integration over a small circle centered around $H' = -b$, see figure 1. This was the contour considered in ref. [16] which is therefore equivalent to the contour $\mathcal{C}$ used in this article. As a consequence, one obtains

$$G(y,0,\chi;y'',0,\chi') = \frac{y''}{2\pi^2\hbar^2\kappa^2}\int_{\mathcal{C}} \mathrm{d}H' \exp\left[\frac{i}{\hbar\kappa}\left(-\frac{2ybH'}{H'+b} + 2y''H'\right)\right]. \tag{3.11}$$

It has been shown in ref. [16] that the integration over H' along the contour $\mathcal{C}$ can then be done explicitly. The exact result is expressed in eq. (4.25) of ref. [16] in terms of a Bessel function of order one. We do not present this result here since this aspect of the calculation has been given in details in ref. [16].

Hence, we have established the following result: there is no quantum change of topology from or to a flat spacelike hypersurface in the homogeneous background, at least at second order in the long-wavelength approximation.

3.3 The case of curved hypersurfaces

We now assume that $v'' \neq 0$. As discussed in the previous subsection, this implies that R and R' are non-vanishing. This time, the integration over J', in the right-hand-side of eq. (3.9), gives the following Dirac function

$$\delta\left[v\left(\frac{b}{H'+b}\right)^{2/3} - v''\right]. \tag{3.12}$$

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Since H' is real, the argument of the Dirac function has zeros if and only if R and R' (or, equivalently, v and v'') have the same sign and, in this case, the zeros are given by

$$H' = H'^{\mp}_0 \equiv -b\left[1 \mp \left(\frac{v}{v''}\right)^{3/2}\right]. \tag{3.13}$$

If R and R' have not the same sign, the corresponding Green function is zero and quantum topology change is forbidden. Therefore, as in the flat case, there is no quantum change of topology.

Despite this fact, and for the sake of completeness, we now present the resulting Green function since this represents a non trivial generalization of the results obtained in ref. [16]. Let us notice that this case is different from the one given in eq. (3.10). There, we had a Dirac function of v'' only and, therefore, it was possible to put the Dirac function outside the integral over H'. Here, we face a different situation since, this time, the Dirac function does depend on H' (the second flat case , i.e. $v \neq 0$, also presented this feature but since the argument of the Dirac function had no zeros, this case was in fact trivial). In order to calculate the integral (3.9) for the present case, we express the Dirac function as

$$\delta\left[v\left(\frac{b}{H'+b}\right)^{2/3} - v''\right] = \frac{3b}{2|v|}\left(\frac{v''}{v}\right)^{-5/2} \times \left[\delta(H' - H'^{-}_0) + \delta(H' - H'^{+}_0)\right], \tag{3.14}$$

and the integration becomes trivial. As a consequence, the Green function is given by the sum of two contributions coming from the two Dirac functions in eq. (3.14)

$$G_r = (G_+ + G_-)\Theta\left(\frac{\mathcal{K}}{\mathcal{K}''}\right), \tag{3.15}$$

where Θ is the step function and where $G_\pm$ are given by

$$\begin{aligned} G_+ &= \frac{3my''x^{5/2}}{\pi\hbar\kappa \mid v \mid (\chi - \chi'')} \exp\left\{\frac{i}{\hbar\kappa}\left[-\frac{2yb(x^{3/2}-1)}{x^{3/2}} + 2y''b(x^{3/2}-1) + \right.\right. \\ &\qquad \left.\left. + \frac{\mathcal{K}y^{1/3}(x^{5/2}-1)}{5b(x^{3/2}-1)x}\right]\right\}, \qquad (3.16) \\ G_- &= \frac{3my''x^{5/2}}{\pi\hbar\kappa \mid v \mid (\chi - \chi'')} \exp\left\{\frac{i}{\hbar\kappa}\left[-\frac{2yb(x^{3/2}+1)}{x^{3/2}} - 2y''b(x^{3/2}+1) + \right.\right. \\ &\qquad \left.\left. + \frac{\mathcal{K}y^{1/3}(x^{5/2}+1)}{5b(x^{3/2}+1)x}\right]\right\}, \qquad (3.17) \end{aligned}$$

with $x \equiv (y/y'')^{1/3}$. This completes our study of the Green functions in the case of real metrics. Note that $G_\pm$ satisfy the Wheeler-DeWitt equation which, at the semiclassical level, reads ($\Psi = Re^{iS}$)

$$m\frac{\partial S}{\partial \chi} - \frac{3\kappa}{4}y\left(\frac{\partial S}{\partial y}\right)^2 - \frac{\mathcal{K}y^{1/3}}{2\kappa} = 0, \tag{3.18}$$

both for y and χ as well as for y' and χ' (taking care that, for χ', the time direction must be reversed).

We have demonstrated that there is no quantum change of topology in that case, as long as this transition involves a change of sign in the curvature of the spatial sections. Our formulation does not allow to signal a change of topology when the transition involves compactified spacelike sections with distinct topology but having the same sign of curvature, a fact that would require a completely distinct and much more elaborated approach to the problem. These conclusions are valid up to second order in the gradient expansion.

The result presented in this section is admittedly expected and rather trivial. Indeed, the phase of the wave function being the classical action, it is clearly impossible to obtain a non-classical dynamics. In this case, no degeneracies in the metric and/or changes of signature are allowed and, hence, no topology changes can occur. However, in the next section, we will demonstrate that this situation can be circumvented, in the case of curved hypersurfaces only, by considering the metric to be a complex quantity. This will lead to a non-classical phase and will imply that signature and/or topology changes become possible. For flat hypersurfaces, as just noticed above, the situation will be different. The fact that $v''=0$ is equivalent to $v=0$ is a strong indication that even complex metrics are of no help in this particular case and, hence, we will be able to establish a non-trivial selection rule: there is no topology change between flat and curved hypersurfaces.

4. The Green function with complex metrics

We have just seen in the previous section that there is no change of topology if H' remains real. This is linked to the fact that the equation $v''=v[b/(H'+b)]^{2/3}$ has no solution on the real axis of H' if v and v'' have opposite signs. It is worth recalling again that the extension of H' to the complex plane made in section III, in the case of flat hypersurfaces, was a mere mathematical trick to calculate the integrals and all the poles inside the contours were located on the real axis. On the other hand, the above equation possesses a solution in the complex plane, namely

$$H'=H'_0\equiv -b\left(1+i\left|\frac{v}{v''}\right|^{3/2}\right) \tag{4.1}$$

and, therefore, this strongly suggests to consider H' as a true complex variable since, in this case, a change of topology is possible. As will be discussed in this section, the complexification of H', with its natural counterpart, namely, a complexification of the metric itself, is a physical requirement. This will lead us to interpret topology change as a quantum tunneling effect. In the case of complex metrics, the corresponding Green function needs to be modified and its expression can be obtained from the expressions derived before. We now describe how this can be done.

4.1 General derivation

In this subsection, we discuss the introduction of complex metrics in more details. There are clearly many questions which can be posed with regards to the use of complex metrics. In particular, in this article, we argue that they represent a natural generalization of techniques used in standard quantum mechanics. Indeed, the use of complex trajectories

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(with imaginary spatial coordinates) in order to describe the tunnel effect is standard and has been used elsewhere. The idea of extending the sum to include complex orbits (or classically forbidden orbits) in the construction of Green functions has been successfully used to describe the tunnel effect, for instance in the energy spectra of molecules in atomic physics [18]. Atomic physicists calculate the trace of the energy-dependent Green functions which, for bounded systems, give the energy levels of the hamiltonian, the trace being obtained by a sum over classical periodic orbits. If the sum is extended to complex periodic orbits (or classically forbidden orbits) as well as to real (or classically allowed orbits) the structure of the energy levels is correctly obtained, where shifts due to the tunnel effect are correctly included. This prescription is heuristic and in general no attempt has been made to a more rigorous treatment of the method.

Let us now explain why the introduction of complex metrics is not only crucial due to the necessary complex nature of H', but is also very useful since it allows us to circumvent the complicated *midisuperspace* formulation of the problem described in ref. [9]. We redefine the metric as

$$\mathrm{d}s^2 = -N_{\mathrm{c}}^2(t)\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}z^2 + \sin^2 z(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\right], \tag{4.2}$$

where N_{c} and z are now allowed to be complex. For instance, setting $N_{\mathrm{c}} = N$ and $z = \xi$ to be real yields, let us say for the initial configuration, a positively curved spatial hypersurface in a lorentzian four-geometry, while requiring for the final configuration that $N_{\mathrm{c}} = -iN$ and $z = -i\xi$ are purely imaginary yields a negatively curved spatial hypersurface, also in a lorentzian four-geometry. Let us notice that there is an overall change of sign between the two metrics. We see that a transition from positively to negatively curved spatial sections may be obtained with a mere complexification of the coordinates, without changing the functional form of the metric in eq. (4.2) or without requiring to introduce a time dependence of $\mathcal{K}$, as in eq. (1.1). This complexification of the coordinates is equivalent to a complexification of the geometry in the following way: $g_{00}(\lambda) = -\left[u_0(\lambda)N + iv_0(\lambda)N'\right]^2$, $g_{11}(\lambda) = a^2\left[u_1(\lambda) + iv_1(\lambda)\right]^2$, $g_{22}(\lambda) = a^2\left[u_2(\lambda)\sin\xi + iv_2(\lambda)\sinh\xi\right]^2$ and, finally, $g_{33}(\lambda) = a^2\sin^2\theta\left[u_3(\lambda)\sin\xi + iv_3(\lambda)\sinh\xi\right]^2$, with $u_\mu(\lambda_{\mathrm{i}}) = v_\mu(\lambda_{\mathrm{f}}) = 1$ and $u_\mu(\lambda_{\mathrm{f}}) = v_\mu(\lambda_{\mathrm{i}}) = 0$, where λ_{i}, λ_{f} denote the initial and final parameter along the trajectory while N, N' are the lapse function in the initial and final configuration respectively. The above example is an explicit realization of the idea that a change of topology can be seen as a trajectory $g_{\mu\nu} = g_{\mu\nu}(\lambda)$ in the space of complex geometries. As a result, the two lorentzian metrics corresponding to the final and initial configurations must now be viewed as belonging to the space of complex metrics. In that space, the metric corresponding to the final configuration is on the "*imaginary axis*" while the metric describing the initial configuration is on the "*real axis*". Another consequence is that the transition between the two configurations can now be realized along "complex trajectories" without necessarily passing through real lorentzian geometries with $\mathcal{K} = 0$.

If such a transition is quantum-mechanically allowed, then there must exist a propagator describing the corresponding dynamical evolution (via the Wheeler-DeWitt equation) from one configuration to the other. Once we have the semiclassical solutions of the

Wheeler-DeWitt equation in the long-wavelength approximation for the two configurations (with a "marker of topology change" explicitly present, in our case the three-curvature of the spatial sections), we can immediately obtain the corresponding Green function by applying the following method. One starts from eq. (3.3) and consider the semiclassical wavefunctions $\Psi(y, R, \chi)$ and $\Psi(y', R', \chi')$ corresponding respectively to the final and initial configurations. Then one can find a Green function G that connects these two configurations by using the procedure of ref. [16] from which we have learned how eq. (3.3) can be manipulated in order to isolate G. The Green function should exist since eq. (3.3) is well defined, topology changes are quantum-mechanically allowed and $\Psi(y, R, \chi)$ and $\Psi(y', R', \chi')$ are solutions of the problem. This is straightforward when there is no change of topology and was done in the previous section. In order to implement the above strategy in the case where there is a change of topology, we just have to exhibit two solutions $\Psi(y, R, \chi)$ and $\Psi(y', R', \chi')$, obtained by means of the long-wavelength approximation and corresponding to lorentzian geometries of the type given in eq. (4.2) but with distinct spatial curvatures $\mathcal{K}$. According to the previous section, the wave-function corresponding to $N_{\rm c}$ and a real is given by

$$\Psi(y', R', \chi') = \exp\left[\frac{i}{\hbar\kappa}\left(-2y'H' + v'J'\right)\right]. \tag{4.3}$$

Obtaining the solution corresponding to $N_{\rm c}$ and a imaginary is slightly more complicated and the remaining of the present subsection will be mainly devoted to this task.

Let us start by noticing that, in principle, we should first implement the ADM formalism for the complex metrics of the type given in eq. (4.2), then obtain a solution in the long-wavelength approximation and finally specify the result to the case where the metric is on the "*imaginary axis*". However, we can circumvent this complicated procedure by using the results given in section II but adapted to the case under consideration. In order to do so, the relevant remark is that the change in the lapse function, $N_{\rm c} = -iN$, is equivalent to a Wick rotation to the imaginary time $\tau = -it$, if one understands t as the proper time of observers whose trajectories are orthogonal to the spacelike hypersurfaces. Therefore, the ADM action must be modified because we have, besides the change $\xi \to -i\xi$, a change to imaginary time which must be treated with great care in order not to make signs mistakes in the hamiltonian formalism. It is also important to realize that we are not dealing with "pure" euclidean quantum gravity. The reason is that there is not only a change to the imaginary time but also to imaginary space. In fact, we consider a transition between two lorentzian geometries with an overall change of signature while, in euclidean quantum gravity, we would have either a transition from an euclidean metric to a lorentzian metric [7], or "instantons", connecting two lorentzian geometries with the same overall signature [5, 6].

Let us first consider the transition to imaginary time. The ADM action defined as $I \equiv -iS[g_{\rm W}]$, see ref. [19], is given by

$$I = -\int {\rm d}^4x\,\sqrt{g_{\rm W}}\left[\frac{1}{2\kappa}{}^{(4)}R(g_{\rm W}) - \frac{n}{2m}\left(g_{\rm W}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + m^2\right) - V(\chi)\right], \tag{4.4}$$

where g_{W} denotes the 4-geometry after the Wick rotation to imaginary time the line element of which can be expressed as

$$\mathrm{d}s^2 = (N^2 + \gamma^{ij} N_i N_j)\mathrm{d}t^2 + 2N_i\, \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^i\,. \tag{4.5}$$

Notice that we have used the relation $\sqrt{-g_{\mathrm{c}}} = -i\sqrt{g_{\mathrm{W}}}$, where g_{c} is a lorentzian metric complexified as in eq. (4.2). Having determined what the action is, we can study the hamiltonian formalism. Calculations very similar to those performed in the standard ADM formalism show that the hamiltonian obtained from the Wick rotated action given above reads

$$H_{\mathrm{W}} = N\mathcal{H}_{\mathrm{W}} + N^i\mathcal{H}_i\,, \tag{4.6}$$

where $\mathcal{H}_{\mathrm{W}}$ and $\mathcal{H}_i$ are respectively given by

$$\begin{aligned}\mathcal{H}_{\mathrm{W}} &= -\kappa\, \gamma^{-1/2}\pi^{ij}\pi^{k\ell}(2\gamma_{jk}\gamma_{\ell i} - \gamma_{ij}\gamma_{k\ell}) - \frac{1}{2\kappa}\gamma^{1/2}R + \\ &\quad + i(m^2 + \chi_{,j}\chi^{,j})^{1/2}\pi^{\chi} + \gamma^{1/2}V(\chi)\,, &(4.7)\\ \mathcal{H}_i &= -2(\gamma_{i\ell}\pi^{\ell k})_{,k} + \pi^{\ell k}\gamma_{\ell k,i} + \pi^{\chi}\chi_{,i}\,, &(4.8)\end{aligned}$$

and where, as before, π^{ij} are the conjugate momenta to γ_{ij}, and π^{χ} is the conjugate momentum to the dust field χ. The quantity R is the Ricci scalar of the three-metric γ_{ij}. From the expression of $\mathcal{H}_{\mathrm{W}}$ derived above, one can establish the Wick rotated Hamilton-Jacobi equation. It reads

$$\begin{aligned}&-\gamma^{-1/2}\kappa\frac{\delta S}{\delta\gamma_{ij}(x)}\frac{\delta S}{\delta\gamma_{k\ell}(x)}[2\gamma_{i\ell}(x)\gamma_{jk}(x) - \gamma_{ij}(x)\gamma_{k\ell}(x)] + \\ &\quad + i\sqrt{m^2 + \gamma^{ij}\chi_{,i}\chi_{,j}}\frac{\delta S}{\delta\chi(x)} + \gamma^{1/2}\ V(\chi) - \frac{1}{2\kappa}\ \gamma^{1/2}R = 0\,, \end{aligned}\tag{4.9}$$

while the momentum constraint equation remains the same as in eq. (2.10). Comparing eq. (4.9) with eq. (2.9), one can see that going to the imaginary time (by means of the Wick rotation) changes the sign of the kinetic term in the gravitational sector. This is a well-known modification when going from lorentzian to euclidean signatures after a Wick rotation. On the other hand, the matter sector, containing the term $\delta S/\delta\chi$, is multiplied by complex number i. This latter change was also expected since χ plays the role of time. Setting $\tau = -it$ implies $\chi_{\mathrm{c}} = -i\chi$ after the Wick rotation, hence the above result. At this point, the spatial metric γ_{ij} still remains complexified and, from now on, will be denoted with a subscript "c". More generally, quantities with the index "c" will refer to this metric while quantities without this index will refer to the real metric obtained after performing the substitution $z = -i\xi$.

Having the Hamilton-Jacobi equation and the momentum equation at our disposal, we now seek solutions using the gradient expansion. As it was the case before, see eq. (2.12), the invariance by reparametrization suggests a solution of the form

$$S_{\mathrm{c}}^{(0)} = -\frac{2}{\kappa}\int \mathrm{d}^3x\gamma_{\mathrm{c}}^{1/2}H_{\mathrm{c}}(\chi)\,. \tag{4.10}$$

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Inserting this form into the Wick rotated Hamilton-Jacobi equation yields, at the zero order, $H_c^2 = (2im/3)\partial H_c/\partial\chi$. This equation should be compared with the equation written after eq. (2.12). The difference between those two equations is a factor $-i$ as expected. The corresponding solution can be written as

$$H_c = -iH\,, \tag{4.11}$$

where $H(\chi) = 2m/[3(\chi - \tilde{\chi})]$. Again, this solution bears a close resemblance with the solution of eq. (2.13).

At second order, the diffeomorphism invariant solution reads

$$S_c^{(2)} = \frac{1}{\kappa}\int \mathrm{d}^3x\gamma_c^{1/2} J_c(\chi) R_c\,, \tag{4.12}$$

and is similar to the solution of eq. (2.17). Applying the same method as before, i.e. inserting the above equation into the Wick rotated Hamilton-Jacobi equation, yields the following relation

$$-H_c J_c + im\frac{\partial J_c}{\partial\chi} = iHJ_c + im\frac{\partial J_c}{\partial\chi} = \frac{1}{2}\,. \tag{4.13}$$

Since we have already established the expression of H_c, the above relation is a differential equation for the quantity J_c only. The solution is very similar to the one found in section II of this paper. It reads

$$J_c = -iJ\,, \tag{4.14}$$

where, as can be deduced from eq. (2.17), one has

$$J(\tilde{\chi}, D) = \frac{3}{10m}(\chi - \tilde{\chi}) + D(\chi - \tilde{\chi})^{-2/3}\,. \tag{4.15}$$

We now have at our disposal the action at zeroth and second order for any metric tensor of the form (4.2) with $N_c = -iN$ real. This allows us to perform the integrals (4.10) and (4.12) exactly. However one has to pay attention to the following fact. When the transformation $z = -i\xi$ is made, the three-metric gets modified by an overall sign (besides the change from sin to sinh). This implies that $\gamma_c^{1/2} = -i\gamma^{1/2}$ and, hence, the volume y_c can be written as

$$y_c \equiv \int \mathrm{d}^3x\gamma_c^{1/2} = -iy\,. \tag{4.16}$$

For the same reason, the three-curvature satisfies $R_c \equiv R(\gamma_c^{ij}) = -R(\gamma^{ij}) \equiv -R$. Combining the previous results, one has

$$v_c = y_c R_c = iyR = iv\,. \tag{4.17}$$

We are now in a position where we can calculate the semi-classical wave function in the long wavelength approximation at the time χ. The result reads

$$\Psi(y_c, R_c, \chi) = \exp\left[\frac{i}{\hbar\kappa}(-2y_c H_c + v_c J_c)\right] = \exp\left[\frac{i}{\hbar\kappa}(2yH + vJ)\right]. \tag{4.18}$$

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Equations (4.17) and (4.18) are the main results of this subsection. We have reached our goal since eq. (4.18) is a solution of the Wheeler-De Witt equation in the long-wavelength approximation and for metrics on the "*imaginary axis*" in the space of complex metrics of the type given in eq. (4.2). This solution should be compared with eq. (3.1). One notices that there is a difference, namely a change of sign in the first term of the exponential. This is a consequence of the overall change of signature occurring during the transition. As we are going to show below, this simple change of sign turns out to be crucial in order to obtain a Green function which allows a quantum change of topology and is a solution of the Wheeler-De Witt equation. This also indicates that topology change is possible only with such an overall change of signature, which may be typical of such transitions, as suggested in ref. [19].

Finally, using the above result, the equivalent of eq. (3.3) reads (from now on, we will assume that the complex metric is the initial one)

$$\Psi(y,R,\chi) \equiv \int_0^{\infty} \mathrm{d}y' \int_0^{\infty} \mathrm{d}R'_c G(y,R,\chi;y'_c,R'_c,\chi')\Psi(y'_c,R'_c,\chi') + \\ + \int_0^{\infty} \mathrm{d}y' \int_{-\infty}^{0} \mathrm{d}R'_c G(y,R,\chi;y'_c,R'_c,\chi')\Psi(y'_c,R'_c,\chi')\,. \tag{4.19}$$

From this expression, it is clear that the Green function (4.19) describes the transition between two lorentzian metrics with an overall change of signature and different curvatures (hence different topologies).

The last step is to calculate the Green function explicitly. This Green function describes a transition from a situation where the wave function is expressed by the standard formula, eq. (3.1), to a case where the wave function is modified by the above-mentioned change of sign, see eq. (4.18) (or vice versa if one wishes to consider the opposite transition). As a consequence, in order to isolate the Green function, we just need to repeat the calculations performed between eq. (3.3) and eq. (3.7), bearing in mind that, in the initial configuration, the wave function is now described by eq. (4.18). We obtain the following result

$$G_i(y,R,\chi;y',R',\chi') = \frac{y'}{2\pi^2\hbar^2\kappa^2}\int_{\mathcal{C}} \mathrm{d}H' \int_{-\infty}^{\infty} \mathrm{d}J' \times \tag{4.20}$$
$$\times \exp\left\{\frac{i}{\hbar\kappa}\left[-2yH - 2y'H' - v'J' + \frac{3v(\chi-\tilde{\chi})}{10m} + \frac{vD}{(\chi-\tilde{\chi})^{2/3}}\right]\right\}.$$

At this point, one should compare the above result with eq. (3.9). Being given the previous considerations, it does not come as a surprise that the sign of the second term in the exponential is different. The origin of the difference is clearly the modified wave function of eq. (4.18). Apart from this (crucial) difference, the previous expression is identical to eq. (3.9).

4.2 Explicit calculation of the Green function

Equipped with the modified Green function (4.20), we now return to our main subject, *i.e* the study of the quantum topology changes. We now consider the case where v and v' have opposite signs, namely sign(v)= $-$sign(v').

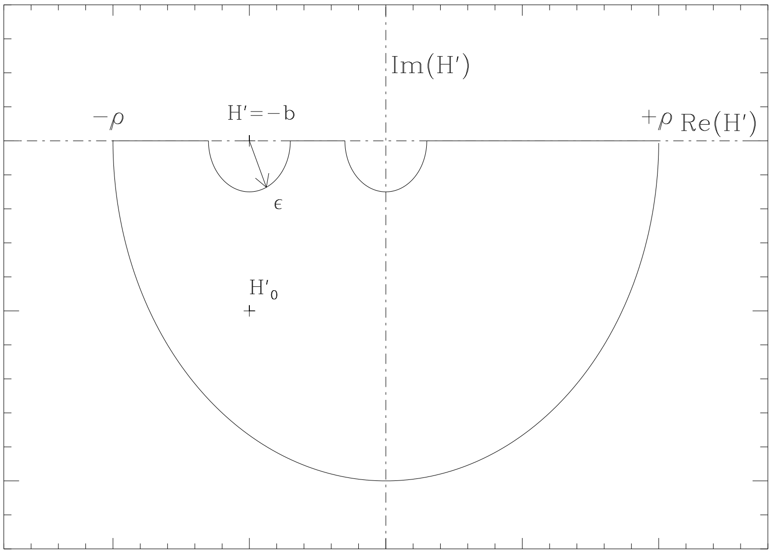

Figure 2: The contour of integration $\mathcal{C}$ in the complex plane $[\Re(H'), \Im(H')]$ in the case of curved hypersurfaces for the modified Green function. The contour chosen is the solid line contour which contains the complex pole of the Dirac function $H_0^{'}$ but avoids the poles of the exponential function, namely $H' = 0$ and $H' = -b$.

The first step is to perform the integration over J'. This yields the Dirac δ-function of eq. (3.12) (except that one should replace v'' by v'). This function has the root given in eq. (4.1), i.e. $H' = H_0'$. The second step is the choice of the contour $\mathcal{C}$. In the complex plane, integration of a Dirac δ-function can be represented by an integration along a small circle around the roots of its argument. For this reason, the integration over H' will be performed along the contour which goes from $-\rho$ to ρ on the real axis, avoiding the poles of the exponential function, $H' = -b$ and $H' = 0$. The contour is closed by considering a large semicircle in the half part of the complex plane with negative imaginary part. This contour is represented in figure 2. It is equivalent to a small closed contour around the pole $H' = H_0'$ because of the Cauchy theorem. In addition, as $\rho \to \infty$, the integral along the large lower semicircle goes to zero and we are reduced to an integral from $-\infty$ to $+\infty$, as before. As announced above, this last argument justifies the fact that we have extracted the Green function from eq. (4.20). Performing explicitly the integration, the Green function for topology change reads

$$\begin{aligned} G_i = {} & \frac{3my'x^{5/2}}{4\pi\hbar\kappa \mid v \mid (\chi - \chi')} \exp\left\{\frac{i}{\hbar\kappa}\left[-\frac{4m(y-y')}{3(\chi-\chi')} + \frac{3\mathcal{K}y^{1/3}(x^4-1)(\chi-\chi')}{10m(x^3+1)x}\right]\right\} \times \\ & \times \exp\left\{\frac{1}{\kappa}\left[-\frac{8m(yy')^{1/2}}{3(\chi-\chi')} - \frac{3\mathcal{K}y^{1/3}x^{1/2}(x+1)(\chi-\chi')}{10m(x^3+1)}\right]\right\}\Theta\left(-\frac{\mathcal{K}}{\mathcal{K}'}\right), \end{aligned} \tag{4.21}$$

where the quantity x has been introduced before. This is the main result of this section.

Let us now check that the above expression possesses the correct properties of a Green function. The Wheeler-DeWitt equation for the homogeneous background is given by

$$-im\hbar\frac{\partial\Psi}{\partial\chi} = -\frac{3\kappa}{4}\hbar^2 y\frac{\partial^2\Psi}{\partial y^2} + \frac{\mathcal{K}y^{1/3}}{2\kappa}\Psi\,. \tag{4.22}$$

The Green function G_i must satisfy the semi-classical version of eq. (4.22). Recovering the units, G_i has the following form: $G_i = C\exp(iS/\hbar)\exp(A/\hbar)$. After inserting this expression in eq. (4.22), one obtains an equation whose real and imaginary parts, in the limit $\hbar \to 0$, read

$$m\frac{\partial S}{\partial \chi} - \frac{3\kappa}{4}y\left[\left(\frac{\partial S}{\partial y}\right)^2 - \left(\frac{\partial A}{\partial y}\right)^2\right] - \frac{\mathcal{K}y^{1/3}}{2\kappa} = 0\,, \tag{4.23}$$

$$-m\frac{\partial A}{\partial \chi} + \frac{3\kappa}{2}y\frac{\partial S}{\partial y}\frac{\partial A}{\partial y} = 0\,. \tag{4.24}$$

Using the expression of G_i derived above, see eq. (4.21), one can easily check that eqs. (4.23) and (4.24) are indeed satisfied in the long-wavelength approximation. Let us notice that terms involving $R^2 = (6\mathcal{K}/y^{2/3})^2$ have been neglected. In order to verify that G_i does indeed satisfy the Wheeler-DeWitt equation at the semiclassical level, one must remember that $y'_{\rm c} = -iy'$ and that, in the case of an imaginary time, the Wheeler-DeWitt is derived from the hamiltonian constraint given in eq. (4.7) which yields (with χ' time reversed):

$$-im\frac{\partial S}{\partial \chi'} + \frac{3\kappa}{4}y'_{\rm c}\left[\left(\frac{\partial S}{\partial y'_{\rm c}}\right)^2 - \left(\frac{\partial A}{\partial y'_{\rm c}}\right)^2\right] - \frac{\mathcal{K}'{y'_{\rm c}}^{1/3}}{2\kappa} = 0\,, \tag{4.25}$$

$$im\frac{\partial A}{\partial \chi'} - \frac{3\kappa}{2}y\frac{\partial S}{\partial y'_{\rm c}}\frac{\partial A}{\partial y'_{\rm c}} = 0\,. \tag{4.26}$$

Taking $y'_{\rm c} = -iy'$, we obtain

$$m\frac{\partial S}{\partial \chi'} - \frac{3\kappa}{4}y'\left[\left(\frac{\partial S}{\partial y'}\right)^2 - \left(\frac{\partial A}{\partial y'}\right)^2\right] + \frac{\mathcal{K}'{y'}^{1/3}}{2\kappa} = 0\,, \tag{4.27}$$

$$-m\frac{\partial A}{\partial \chi'} + \frac{3\kappa}{2}y\frac{\partial S}{\partial y'}\frac{\partial A}{\partial y'} = 0\,. \tag{4.28}$$

One can check that eq. (4.21) satisfies eqs. (4.27) and (4.28) for $\mathcal{K}' = -\mathcal{K}$, thus demonstrating the change of topology.

Hence, G_i possesses all the good properties of a Green function signaling a topology change: it describes the transition from a wave function describing an homogeneous spatial geometry with positive (negative) curvature to another wave function describing another homogeneous spatial geometry with negative (positive) curvature [see eq. (3.3)], and it satisfies the semi-classical Wheeler-DeWitt equation in the long-wavelength approximation.

4.3 Discussion

In this section, we briefly discuss the main properties of the Green function of eq. (4.21). First of all, it is interesting to notice that, without the change of sign obtained in the first term of the exponential, see eq. (4.18), the Green function obtained through eq. (3.3) would not satisfy the Wheeler-DeWitt equation (4.22). This change of sign is a direct consequence of describing the change of topology by means of complexification of metrics , which in turn implies an overall change of sign in the four-metric as the transition takes place. Therefore, in this framework, topology changes without an overall change of signature are not possible. We notice in passing that eq. (4.23) is not exactly the classical Hamilton-Jacobi equation for a *minisuperspace* corresponding to the homogeneous metrics considered here. There is an extra term which induces the quantum change of topology. This is the quantum potential in the Bohm-de Broglie interpretation language. This extra term comes from the imaginary part A of the action and indicates that such a change of topology is in fact a quantum tunneling effect.

Secondly, examining the Green function (4.21), one can notice that a topology change is very improbable if $\Delta\chi \equiv \chi - \chi'$ is very short. As $\Delta\chi$ increases, quantum changes of topology become possible in both directions (from positive to negative curvature and vice-versa), but when $\Delta\chi$ becomes very large, negative to positive curvature transitions are suppressed while positive to negative curvature transitions are enhanced. This suggests that negative curvature topologies are preferred. Finally, for finite $\Delta\chi$, topology transitions between large volume spacelike hypersurfaces are very improbable, as expected.

5. Conclusions

We now briefly summarize the main results reached in this article. We have obtained a Green function describing a topology change by using a double approximation: the semi-classical canonical quantum gravity and the long wavelength approximation. Our considerations have been restricted to transitions between homogeneous spacelike hypersurfaces with different intrinsic curvatures. In this framework, we have been able to demonstrate that transitions involving flat hypersurfaces are forbidden. However, we have also shown that this is no longer the case if the hypersurfaces are curved. In order to obtain the corresponding Green functions describing transitions between curved hypersurfaces, we have used imaginary coordinates (not only timelike coordinate but also one spacelike coordinate) in order to circumvent the *midisuperspace* problem raised in ref. [9] (i.e. the components of the four-dimensional curvature tensor are only functions of time). The explicit expression of the Green function has been obtained by performing an integration over all possible initial configurations in the complex plane. The resulting Green function shows that topology changes in the direction of negatively curved hypersurfaces are strongly enhanced as time goes on while transitions in the opposite direction are suppressed. In a finite amount of time, transitions between large spacelike hypersurfaces are improbable.

The formalism developed in this paper can also be applied to more complicated models. This will be the subject of our future investigations.

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